Project NExT Workshop
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Teaching Real Analysis—An active approach

Activity 1: Below you will find several statements involving a sequence (a_n) of real numbers and a real number L. In each case, consider the statement as an "alternative" to the definition of $(a_n) \to L$. Provide an example of a sequence (a_n) of real numbers and a number L that satisfies the "definition" and yet does not converge to L. Accompany your example with a verbal explanation of the inadequacies of the definition.

1. The sequence (a_n) converges to L if for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(a_n, L) < \epsilon$.
2. The sequence (a_n) converges to L if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for some $n > N$, $d(a_n, L) < \epsilon$.
3. The sequence (a_n) converges to L if for all $N \in \mathbb{N}$ there exists $\epsilon > 0$ so that for all $n > N$, $d(a_n, L) < \epsilon$.
4. The sequence (a_n) converges to L if for all $N \in \mathbb{N}$ and all $\epsilon > 0$ there exists $n > N$ such that $d(a_n, L) < \epsilon$.[1]

Activity 2: The second "epsilonics" definition. Your students have been thinking about sequence convergence for a while and now you are to tackle continuity. How do you start with your students' prior understanding of continuity (from calculus) and end up with the standard $\epsilon - \delta$ definition for continuity?

Activity 3: Starting with only the definition of sequence convergence, prove that the real sequence $0, 1, 0, 1, 0, 1, \ldots$ does not converge. Once again, psychoanalyze yourselves and your students in this mathematical situation.

Activity 4: Work on proofs of the following three standard theorems. Ask yourselves what kinds of "skills and practices" you are using, what "presuppositions or assumptions" you are bringing to bear, and how you know "where to focus your attention."

- Prove that $\lim_{x \to 3} \dfrac{x^2}{1+x^2} = \dfrac{9}{10}$.
- Let (a_n) and (b_n) be sequences in $\mathbb{R}$. Suppose that $a_n \to L$ and $b_n \to M$. Prove that $(a_n b_n)$ converges to LM.

[1]This exercise is taken from *Closer and Closer: Introducing Real Analysis* by Carol S. Schumacher, ©Jones and Bartlett Publishers, 2008.

- Let X and Y be metric spaces, and let $f : X \to Y$ be a function. Prove that f fails to be uniformly continuous if and only if there exist $\epsilon > 0$ and sequences (x_n) and (y_n) in S such that $d_X(x_n, y_n) \to 0$ and yet for all $n \in \mathbb{N}$, $d_Y(f(x_n), f(y_n)) \geq \epsilon$

Activity 5: You are teaching a real analysis class and have just defined continuity. Your students have been told that K is a fixed real number, that x is a fixed element of the metric space X and that $f : X \to \mathbb{R}$ is a continuous function. They have been asked to prove that if $f(x) > K$, then there exists an open ball about x such that f maps every element of the open ball to some number greater than K.

One of your students comes into your office saying that he has "tried everything" but cannot make any headway on this problem. When you ask him what exactly he has tried, he simply reiterates that he has tried "everything." What is happening? What do you do?

Activity 6: Work on proofs of the following three standard theorems. Think about pictures. Think about the relationship between analysis and geometry. How does each problem speak to this very important relationship?

- Let K be a non-empty subset of $\mathbb{R}$. We say that K is **bounded** if K is bounded both above and below. That is, if there exist real numbers m and M such that for all $k \in K$, $m \leq k \leq M$.

  Prove that K is bounded if and only if there exists a real number T such that for all $k \in K$, $|k| \leq T$.

- Let B be a non-empty subset of $\mathbb{R}$ that is bounded above (and therefore has a supremum/least upper bound, $\sup B$.) Let $b \in \mathbb{R}$ be an upper bound for B. Prove that the following statements about b are equivalent.

  1. $b = \sup B$.
  2. For each positive number ϵ there exists $x \in B$ such that $|x - b| < \epsilon$.
  3. For each positive number ϵ there exists $x \in B$ such that $x \in (b - \epsilon, b]$.

- Let K be a subset of $\mathbb{R}$. Let $f : K \to \mathbb{R}$ be a one-to-one, continuous function. Show that the inverse function $f^{-1} : f(K) \to \mathbb{R}$ need not be continuous. Find a hypothesis on K that is sufficient to guarantee the continuity of f^{-1}.